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A Generalized Quasiaverage Approach to the Description of the Limit States of the *n*-Vector Curie–Weiss Ferromagnet

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A set of all limit (Gibbs) states is constructed for the ferromagnetic n-vector Curie–Weiss model by means of a generalized quasiaverage method.

KEY WORDS: Limit states; pure and mixed phases; quasiaverage method; inhomogeneous external fields.

1. INTRODUCTION

Bogolubov's quasiaverage method[1] appeared as a means of removing the degeneracy (i.e., of separating the pure phases) in statistical-mechanical models exhibiting phase transitions accompanied by spontaneous symmetry breaking. It consists in switching on a source term (external field) in the Hamiltonian which reduces its symmetry to that of the corresponding pure phase below the transition point; after the thermodynamic limit of the correlation functions has been calculated, the source is switched off. This approach allowed the construction of the exact solution of the BCS-model[2] and other mean-field models; see, e.g., Refs. 3 and 4. It is now an important tool in the study of models with spontaneous symmetry breaking; see, e.g., Refs. 5–7.

At present, the description of the infinite-volume equilibrium states of models in classical statistical mechanics makes use of the general notion of limit Gibbs measure, as first introduced by Minlos.[8] These were defined in

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the whole generality for systems with bona fide interactions as measures with prescribed conditional expectations by Dobrushin[9] and Lanford and Ruelle[10] (DLR equations); see also Refs. 11 and 12. It turns out that for such systems the whole set of the limit Gibbs measures coincides with the set of all weak limits of finite-volume Gibbs measures subject to various boundary conditions.

On the other hand, for models of the mean-field type, where the interaction depends on volume and where is no notion such as interaction in the infinite system, one cannot define limit Gibbs measures via DLR equations. The set of infinite-volume equilibrium states should be obtained as weak limits of finite-volume Gibbs states.[8]

In this note we construct for the n-vector Curie–Weiss model infinite-volume equilibrium states (or limit states) as limits of the finite-volume Gibbs states. To this end, we propose in Section 2 a generalization of the Bogolubov's quasiaverage method consisting in the following: small external symmetry-breaking fields switched on in a finite volume are allowed to be inhomogeneous and to depend on volume; also (as we are interested in zero-field states where multiple phases arise) they are allowed to go to zero simultaneously with the volume going to infinity. By this generalization one should obtain not only the pure phases, but all equilibrium states including possibly nontranslation invariant states and mixed states.

Though the Curie–Weiss model has been extensively studied from very different points of view (see, e.g., Refs. 13–15), as far as we know, in the literature there exists no attempt of a complete description of all its equilibrium states, except a recent paper[16] on Curie–Weiss–Ising model. It turns out that for the n-vector Curie–Weiss model: (i) all limit states we obtain by the above procedure possess the whole symmetry of the zero-field Hamiltonian under the permutations of spins; (ii) these states are convex combinations of pure phases (as constructed by the usual quasiaverage method). More precisely, for the n-vector Curie–Weiss model in zero external field the limit states are completely specified by the average magnetization, and thus they correspond to the points of the ball in $\mathbb{R}^n$ of radius equal to the spontaneous magnetization: the pure states are the points on the sphere, whereas the mixed states correspond to interior of the ball. Section 3 is devoted to limit states in nonzero inhomogeneous external fields. It is shown that for almost all (with respect to a product probability measure) external field configurations the limit state exists and is determined by one "average" self-consistency equation.

2. LIMIT STATES AT ZERO EXTERNAL FIELD

We start with defining the ferromagnetic Curie–Weiss model in an external field for N spins. Let $\mathbf{h}_N^{(N)} = (\vec{h}_1^{(N)},..., \vec{h}_N^{(N)}) \in \mathbb{R}^{Nn}$ and $\beta > 0$ be given. To every $j = 1,..., N$ we assign a spin, i.e., a random vector in $\mathbb{R}^n$ of unit length: $\vec{\sigma}_j \in S^{n-1}$; the unit sphere S^{n-1} is endowed with the rotationally invariant probability measure $d\vec{\sigma}$. The joint probability distribution of $\boldsymbol{\sigma}_N = (\vec{\sigma}_1,..., \vec{\sigma}_N)$ corresponding to the external fields $\mathbf{h}_N^{(N)}$ and temperature β^{-1} is given according to the Gibbs prescription by the following density with respect to the free product measure $d\boldsymbol{\sigma}_N = \prod_{j=1}^N d\vec{\sigma}_j$:

$$p_N(\boldsymbol{\sigma}_N; \beta, \mathbf{h}_N^{(N)}) = [Z_N(\beta, \mathbf{h}_N^{(N)})]^{-1} \exp[-\beta \mathscr{H}_N(\boldsymbol{\sigma}_N, \mathbf{h}_N^{(N)})] \tag{1}$$

where $Z_N(\beta, \mathbf{h}_N^{(N)})$ is the partition function and the Hamiltonian is

$$\mathscr{H}_N(\boldsymbol{\sigma}_N, \mathbf{h}_N^{(N)}) = -\frac{1}{2N}\left(\sum_{j=1}^N \vec{\sigma}_j\right)^2 - \sum_{j=1}^N \vec{\sigma}_j \cdot \vec{h}_j^{(N)} \tag{2}$$

From (1) one obtains the distribution of k spins, say, $\vec{\sigma}_1,..., \vec{\sigma}_k$, by integrating over the remaining spins:

$$P_k^{(N)}(d\vec{\sigma}_1,..., d\vec{\sigma}_k; \beta, \mathbf{h}_N^{(N)}) = \left\{\int_{(S^{n-1})^{N-k}} \prod_{j=k+1}^N d\vec{\sigma}_j\, p_N(\boldsymbol{\sigma}_N; \beta, \mathbf{h}_N^{(N)})\right\} d\boldsymbol{\sigma}_k \tag{3}$$

Suppose now that the fields $\mathbf{h}^{(N)}$ have been chosen such that for every fixed k the following limit exists:

$$P_k(d\vec{\sigma}_1,..., d\vec{\sigma}_k; \beta, \mathbf{h}) = \lim_{N\to\infty} P_k^{(N)}(d\vec{\sigma}_1,..., d\vec{\sigma}_k; \beta, \mathbf{h}_N^{(N)}) \tag{4}$$

It is easy to check from Eq. (3) that $\{P_k\}_{k=1}^\infty$ is a consistent family of probability distributions on the configuration space $\mathscr{T} = (S^{n-1})^{\mathbb{N}}$ (a compact with the Borel $\sum$ algebra relative to the product topology we put on $\mathscr{T}$). Then, by Kolmogorov's theorem (see, e.g., Ref. 17), there exists a unique probability measure P on $\mathscr{T}$ whose finite-dimensional distributions (projections) on $\prod_{j=1}^k S^{n-1}$ are P_k.

A probability measure P on $\mathscr{T}$ defined in this way from the Hamiltonians (2) will be called (in accordance with Ref. 8). a Gibbs distribution (state) of the infinite-volume Curie–Weiss n-vector model, or simply a limit state.

Of course, P will depend, in general, on the sequence $\{\mathbf{h}^{(N)}\}$. Whenever $\lim_{N\to\infty} \vec{h}_j^{(N)} = \vec{h}_j$ exists for all $j \in \mathbb{N}$, we associate P with the external field configuration $\mathbf{h} = \{\vec{h}_j\}_{j\in\mathbb{N}}$. In particular, we are interested in limit states corresponding to zero external field ($\vec{h}_j = 0$ for all j) because for

$\mathbf{h}=0$ the Hamiltonian $\mathscr{H}_N$ is invariant under the permutation of the spins and under simultaneous rotation of all spins, while the limit states are not necessarily such. Before stating the result, we fix some notation and briefly remind the calculation of the free energy for the zero-field case, i.e., of $\lim_{N\to\infty}(N\beta)^{-1}\ln Z_N(\beta, 0)$.

Let us define the function φ_n: $[0, \infty)\to[0, \infty)$ by

$$\varphi_n(y)=\beta^{-1}\ln\int_{S^{n-1}} d\vec{\sigma}\exp\beta\vec{\sigma}\cdot\vec{y}, \qquad y=|\vec{y}| \tag{5}$$

This is possible because the right-hand side is obviously rotation invariant. It is clear that $\varphi_n(y)$ is increasing and convex, while $\varphi_n'(y)$ is concave and increases monotonically from 0 to 1. Moreover, $\varphi_n''(0)=\beta/n$. Therefore, the function f_n: $[0, \infty)\to\mathbb{R}^1$ defined by

$$f_n(y)=\frac{1}{2}y^2-\varphi_n(y) \tag{6}$$

has exactly one minimum point $y_0(\beta)\in[0, \infty)$, which equals zero if $\beta\leqslant n$ and equals to a nonzero solution of the equation

$$y=\varphi_n'(y) \tag{7}$$

if $\beta>n$. Now, using the identity

$$\exp\left[\frac{\beta}{2N}(\vec{S}\cdot\vec{S})\right]=\left(\frac{\beta N}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n} d\vec{y}\left[-\beta\left(\frac{1}{2}y^2N+\vec{S}\cdot\vec{y}\right)\right] \tag{8}$$

and the definitions (5), (6), we have for Z_N

$$\begin{aligned} Z_N(\beta, 0)&=\int_{(S^{n-1})^N} d\boldsymbol{\sigma}_N\exp\left[\frac{\beta}{2N}\left(\sum_{j=1}^N\vec{\sigma}_j\right)^2\right]\\ &=\left(\frac{\beta N}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n} d\vec{y}\exp[-N\beta f_n(y)] \end{aligned} \tag{9}$$

Passing to spherical coordinates and integrating over the angular variables, one obtains, applying the one-dimensional Laplace method

$$\lim_{N\to\infty}(N\beta)^{-1}\ln Z_N(\beta, 0)=-f_n(y_0(\beta)) \tag{10}$$

Thus, critical temperature $\beta_c^{-1}=n^{-1}$ and $y_0(\beta)$ is the spontaneous magnetization.

Let us remark that the Curie–Weiss–Ising case corresponds to $n=1$, $S^0=\{-1, +1\}$ and $\varphi_{n=1}(y)=\beta^{-1}\ln ch\beta y$; see Ref. 16.

Proposition 1. Suppose $\|\mathbf{h}_N^{(N)}\|^2=\sum_{j=1}^N (\vec{h}_j^{(N)})^2 \to 0$ as $N\to\infty$ and denote $H_N=|\vec{H}_N|$, $\vec{H}_N=\sum_{j=1}^N \vec{h}_j^{(N)}$ and $\vec{\rho}_N=\vec{H}_N H_N^{-1}$. Then we have the following:

(i) If $H_N\to\infty$ and $\vec{\rho}_N\to\vec{\rho}$ as $N\to\infty$, all the limits (4) exist and equal

$$P_k(d\vec{\sigma}_1,..., d\vec{\sigma}_k; \vec{\rho})=\prod_{j=1}^k p(\vec{\sigma}_j; \vec{\rho})\, d\vec{\sigma}_j, \qquad k=1, 2,... \tag{11}$$

where

$$p(\vec{\sigma}; \vec{\rho})=\exp\{\beta[y_0(\beta)\,\vec{\sigma}\cdot\vec{\rho}-\varphi_n(y_0(\beta))]\} \tag{12}$$

(ii) If $\vec{H}_N\to\vec{H}$ as $N\to\infty$, the limits (4) exist and equal

$$\begin{aligned}\mathscr{P}_k(d\vec{\sigma}_1,..., d\vec{\sigma}_k; \vec{H})=\int_{S^{n-1}} & d\vec{\rho}\,\exp\{-\beta[y_0(\beta)\,\vec{H}\cdot\vec{\rho}+\varphi_n(Hy_0(\beta))]\}\\ & \times P_k(d\vec{\sigma}_1,..., d\vec{\sigma}_k; \vec{\rho}), \quad k=1, 2,...\end{aligned} \tag{13}$$

Proof. We consider first the case $\beta>\beta_c$ when $y_0(\beta)>0$ and prove assertion (ii). Using again the identity (8) and definition (6) one gets for the k-spin distribution density of the finite-volume system [see Eq. (3)] the following expression:

$$\begin{aligned}&p_k^{(N)}(\vec{\sigma}_1,..., \vec{\sigma}_k; \beta, \mathbf{h}_N^{(N)})\\ &\quad=\frac{\int_{\mathbb{R}^n} d\vec{y}\,\exp\{-\beta[\frac{1}{2}Ny^2+\vec{S}_k\cdot\vec{y}-\sum_{j=1}^k \vec{\sigma}_j\vec{h}_j^{(N)}-\sum_{j=k+1}^N \varphi_n(|\vec{y}-\vec{h}_j^{(N)}|)]\}}{\int_{\mathbb{R}^n} d\vec{y}\,\exp\{-\beta[\frac{1}{2}Ny^2-\sum_{j=1}^N \varphi_n(|\vec{y}-\vec{h}_j^{(N)}|)]\}}\end{aligned}$$

where $\vec{S}_k=\sum_{j=1}^k \vec{\sigma}_j$. Developing $\varphi_n(|\vec{y}-\vec{h}|)$ to second order in $\vec{h}$: $\varphi_n(|\vec{y}-\vec{h}|)=\varphi_n(y)-\vec{h}\cdot\vec{y}\varphi_n'(y)\,y^{-1}+R(\vec{y}, \vec{h})$ one obtains

$$\begin{aligned}&p_k^{(N)}(\vec{\sigma}_1,..., \vec{\sigma}_k; \beta, \mathbf{h}_N^{(N)})\\ &\quad=\frac{\begin{aligned}\int_{\mathbb{R}^n} d\vec{y}\,\exp\{-\beta[Nf_n(y)+\vec{H}_N\cdot\vec{y}\varphi_n(y)\,y^{-1}+\vec{S}_k\cdot\vec{y}+k\varphi_n(y)]\}\\ \times\exp\{\beta[\textstyle\sum_{j=1}^k \vec{\sigma}_j\cdot\vec{h}_j^{(N)}+\sum_{j=k+1}^N R(\vec{y}, \vec{h}_j^{(N)})]\}\end{aligned}}{\int_{\mathbb{R}^n} d\vec{y}\,\exp\{-\beta[Nf_n(y)+\vec{H}_N\cdot\vec{y}\varphi_n(y)\,y^{-1}]\}\,\exp\{\beta\sum_{j=1}^N R(\vec{y}, \vec{h}_j^{(N)})\}}\end{aligned} \tag{14}$$

Now the assumption $\vec{H}_N\to\vec{H}$ [see (ii)] implies that $\vec{H}_N\cdot\vec{y}\varphi_n'(y)\,y^{-1}\to$

$\vec{H}\cdot\vec{y}\varphi_n'(y)\,y^{-1}$ uniformly in $\vec{y}$ as $N\to\infty$. On the other hand, using the concavity of $\varphi_n'(y)$ we obtain

$$0\leqslant R(\vec{y},\vec{h})=h^2\varphi_n''(y_\theta)+\left[h^2-\left(\frac{\vec{h}\cdot\vec{y}_\theta}{y_\theta}\right)^2\right]\frac{\varphi_n'(y_\theta)}{y_0}\leqslant 2h^2\varphi_n''(0)=\frac{2\beta}{n}h^2$$

where $\vec{y}_\theta=\vec{y}-\theta\vec{h}$ for some $\theta\in(0,1)$. So, $\|\mathbf{h}_N^{(N)}\|\to 0$ implies that the last exponentials in the numerator and denominator of Eq. (14) converge uniformly in $\vec{y}$ to 1 as $N\to\infty$ Defining the average $\langle-\rangle_1^{(N)}$:

$$\langle g\rangle_1^{(N)}=\frac{\int_{\mathbb{R}^n}d\vec{y}g(\vec{y})\exp[-N\beta f_n(y)]}{\int_{\mathbb{R}^n}d\vec{y}\exp[-N\beta f_n(y)]}$$

one obtains, by the usual Laplace method for each fixed direction $\vec{\rho}$ and then by integrating over the angular variables, that

$$\lim_{N\to\infty}\langle g\rangle_1^{(N)}=\int_{S^{n-1}}d\vec{\rho}\ g(y_0(\beta)\,\vec{\rho})$$

Taking into account that $|\langle g\rangle_1^{(N)}|\leqslant\sup_{\vec{y}}|g(\vec{y})|$ (so that $\langle g_N-g\rangle_1^{(N)}\to 0$ whenever $g_N(\vec{y})\to g(\vec{y})$ uniformly in $\vec{y}\in\mathbb{R}^n$ as $N\to\infty$) we obtain from Eq. (14)

$$\begin{aligned}\lim_{N\to\infty}\ & p_k^{(N)}(\vec{\sigma}_1,\dots,\vec{\sigma}_k;\beta,\mathbf{h}_N^{(N)})\\ &=\lim_{N\to\infty}\frac{\langle\exp\{-\beta[(\vec{H}\cdot\vec{y}/y)\,\varphi_n'(y)+\vec{S}_k\cdot\vec{y}+k\varphi_n(y)]\}\rangle_1^{(N)}}{\langle\exp\{-\beta[(\vec{H}\cdot\vec{y}/y)\,\varphi_n'(y)]\}\rangle_1^{(N)}}\\ &=\frac{\int_{S^{n-1}}d\vec{\rho}\exp\{-\beta[\vec{H}\cdot\vec{\rho}\varphi_n(y_0(\beta))+\vec{S}_k\cdot\vec{\rho}y_0(\beta)+k\varphi_n(y_0(\beta))]\}}{\int_{S^{n-1}}d\vec{\rho}\exp\{-\beta[\vec{H}\cdot\vec{\rho}\varphi_n(y_0(\beta))]\}}\end{aligned}\tag{15}$$

which is identical with Eq. (13) [if we remember that $y_0(\beta)$ is a solution of Eq. (7)]. Thus assertion (ii) is proved.

More care is needed to prove (i) because the angle-dependent part in the exponents in Eq. (14) has also the divergent factor H_N. Defining again an auxiliary probability measure $\langle-\rangle_2^{(N)}$ on $\mathbb{R}^n$ by the density (Q_N is a normalizing factor),

$$Q_N^{-1}\exp\left\{-\beta N\left[f_n(y)+\frac{\vec{H}_N\cdot\vec{y}}{Ny}\varphi_n'(y)\right]\right\}$$

we can discard, as above, the last exponentials in Eq. (14). Thus we need to evaluate only the limit

$$\lim_{N\to\infty} \langle \exp\{-\beta[\vec{S}_k\cdot\vec{y}+k\varphi_n(y)]\}\rangle_2^{(N)}=p_k$$

Taking [with Eq. (5)] the integral over angles, we get

$$p_k=\frac{\int_0^\infty dy\, y^{n-1}\exp\{-\beta[Nf_n(y)-\varphi_n(|\vec{H}_N\varphi_n'(y)+\vec{S}_k y|)+k\varphi_n(y)]\}}{\int_0^\infty dy\, y^{n-1}\exp\{-\beta[Nf_n(y)-\varphi_n(H_N\varphi_n'(y))]\}} \quad (16)$$

Clearly, because $N^{-1}H_N\to 0$ as $N\to\infty$ and the function $\varphi_n(y)$ has a linear growth at infinity, only some compact interval $I\subset(0,\infty)$ containing the minimum of the function $f_n(y)$ will contribute to the limit (16). Let us expand $\varphi_n(|\vec{H}_N\varphi_n'(y)+\vec{S}_k y|)$ for $y\in I$ to the second order in $\vec{S}_k y$:

$$\varphi_n(|\vec{H}_N\varphi_n'(y)+\vec{S}_k y|)=\varphi_n(H_N\varphi_n'(y))+y\vec{\rho}_N\cdot\vec{S}_k\varphi_n'(H_N\varphi_n'(y))+\Delta \quad (17)$$

Then for the remainder Δ [with some $\theta_N(y)\in(0,1)$ and uniformly in $y\in I$] one gets

$$|\Delta|\leqslant\tfrac{1}{2}S_k^2\sup_{y\in I}\{y^2\varphi_n''(|\vec{H}_N\varphi_n'(y)+\vec{S}_k\theta_N(y)\,y|)\}\xrightarrow[N\to\infty]{}0 \quad (18)$$

because derivative $\varphi_n'(y)$ is bounded away from zero on the compact I. As $\varphi_n(y)\to 1$ for $y\to\infty$, we may again apply a uniform convergence argument to obtain from Eqs. (16)–(18):

$$p_k(\vec{\sigma}_1,\dots,\vec{\sigma}_k;\vec{\rho})=\exp\{\beta[y_0(\beta)\,\vec{\rho}\cdot\vec{S}_k-k\varphi_n(y_0(\beta))]\}$$

which proves (11).

For $\beta\leqslant\beta_c$, $y_0(\beta)=0$, and it is neither necessary, nor possible to restrict the integration in (14) over angles to a small interval in a neighborhood of the direction $\vec{\rho}$ [as in Eq. (16)] or with a measure concentrated around $\vec{H}$ [as in Eq. (15)]. One can see that provided $\varepsilon_N>0$ are chosen such that $\varepsilon_N\to 0$ and $\varepsilon_N>(H_N/N)^{1/3}$, then only a neighborhood of the origin of the form $|\vec{y}|^2<\varepsilon_N(|\vec{y}|^4<\varepsilon_N,\beta=\beta_c)$ will contribute to the $N\to\infty$ limit of Eq. (14). On the other hand, in this neighbourhood, $\exp\{-\beta[\vec{S}_k\cdot\vec{y}+k\varphi_n(y)]\}$ approaches (uniformly in angular variables) 1, what coincides with Eqs. (11), (12) [and, of course, with Eq. (13) too] for $y_0(\beta)=0$. ∎

Corollary 1. The measures $\{P_k\}_{k=1}^\infty(\{\mathscr{P}_k\}_{k=1}^\infty)$ on $\mathscr{T}$ satisfy the consistency conditions. Consequently, via Kolmogorov's theorem one can reconstruct on at least two families of the limit states: $dP_{\vec{\rho}}(\boldsymbol{\sigma})$ and $d\mathscr{P}_{\vec{H}}(\boldsymbol{\sigma})$.

The picture of the set of zero-field limit states given in the Proposition and the Corollary is by no means unexpected: it corresponds to the conventional wisdom view. Indeed, if $\beta > \beta_c$ then case (i) describes the pure states (pure phases), which correspond to identical independent distribution of the spins according to the noninvariant distribution $P(d\vec{\sigma}; \vec{\rho})$. In other words, the rotation invariance (but not the permutation invariance) is broken in these states. Independence implies for continuous functions $g_1, g_2 \in C(\mathscr{T})$ with supp $g_1 \cap$ supp $g_2 = \varnothing$ the cluster property:

$$\int g_1 g_2 \, dP_{\vec{\rho}} = \int g_1 \, dP_{\vec{\rho}} \int g_2 \, dP_{\vec{\rho}} \tag{19}$$

which conveys here the purity (ergodicity) of the states $dP_{\vec{\rho}}$. On the other hand, case (ii) describes the mixed states (mixed phases) $d\mathscr{P}_{\vec{H}}$. The measure $d\mathscr{P}_{\vec{H}}$ [see Eq. (13)] is explicitly decomposed into its ergodic components $dP_{\vec{\rho}}$,

$$d\mathscr{P}_{\vec{H}} = \int_{S^{n-1}} \mu_{\vec{H}}(d\vec{\rho}) \, dP_{\vec{\rho}} \tag{20}$$

which are extreme points of the convex set of the limit states, and thus are indecomposable. It has turned out that the probability measure $\mu_{\vec{H}}(d\vec{\rho})$ on S^{n-1} defining the mixed state depends only on the "total magnetic field," $\vec{H}$, see Eq. (13). If $\beta \leqslant \beta_c$, (12) becomes, as expected, the uniform distribution on S^{n-1}, and the distributions (11) and (13) coincide (uniqueness of the limit state above the critical temperature).

Remark 1. If $\{\mathbf{h}_N^{(N)}\}$ is an arbitrary sequence satisfying $\|\mathbf{h}_N^{(N)}\| \to 0$ as $N \to \infty$ (see Proposition 1), then by compactness arguments we can find subsequences $\{N_i\}$ such that $\{\vec{\rho}_{N_i}\}$ converges and $\{H_{N_i}\}$ either converges or diverges to infinity. In both cases we are either in the frame of the case (ii), or of the case (i), and therefore the family $\{P_k^{(N_i)}(d\vec{\sigma}_1,..., d\vec{\sigma}_k; \beta, \mathbf{h}_{N_i}^{(N_i)})\}_{k=1}^{\infty}$ converges for $N_i \to \infty$ to projections of a probability measure on $\mathscr{T}$, i.e., to a limit state.

Remark 2. We view our result as a strong argument in favor of the fact that the Hamiltonian (2) has no inhomogeneous limit states at zero external field if, e.g., $j \in \mathbb{Z}^d$. In order to obtain inhomogeneous states, one has to modify the interactions so as to allow several mean fields as done, e.g., in Kac–Helfand models[18]; thereby interesting phenomena related to nonhomogeneity do appear.[19]

Remark 3. The following observations conserning the assumption $\|\mathbf{h}_N^{(N)}\| \to 0$ for $N \to \infty$ are in order here. This is a regularity condition on the convergence to zero of the external fields. Technically, it allows a simple proof of the Proposition 1. In fact, we claim that the main results: (1) specifying the set of pure states as well as the fact (2) that all other limit states are superpositions of the latter, are valid under the weaken assumption: $\max_{1 \leqslant j \leqslant N} |\vec{h}_j^{(N)}| \to 0$ for $N \to \infty$. For example, there is no difficulty to verify this for homogeneous fields: $\vec{h}_j^{(N)} = \vec{h}^{(N)}$; $j = 1, 2, ..., N$; $|\vec{h}^{(N)}| = cN^{-\alpha}$; one obtains the statement corresponding to case (i) for *all* $0 < \alpha < 1$. Though $\|\mathbf{h}_N^{(N)}\| = cN^{1-2\alpha} \to 0$ only for $\alpha > 1/2$; case (ii) is obtained for $\alpha \geqslant 1$.

As a particular case of certain physical significance, we point out the construction of limit states via "boundary conditions." If the configuration of the last $N - m_N$ spins is frozen: $\vec{\sigma}_j = \vec{s}_j$, $j = m_N + 1, m_N + 2, ..., N$, then the probability distribution of the first m_N spins is given by the Hamiltonian

$$\mathscr{H}_N = -\frac{1}{2N}\left(\sum_{j=1}^{m_N} \vec{\sigma}_j\right)^2 - \left(\frac{1}{N}\sum_{j=m_N+1}^{N} \vec{s}_j\right) \cdot \sum_{j=1}^{m_N} \vec{\sigma}_j \tag{21}$$

Therefore, it is equivalent to a homogeneous external field $\vec{h}^{(N)} = N^{-1} \sum_{j=m_N+1}^{N} \vec{s}_j$ and to a relative shift $N^{-1}(N - m_N)$ in β.

Remark 4. To construct limit states by a conventional quasiaverage procedure, one should calculate the following limits ($k = 1, 2, 3, ...$):

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} P_k^{(N)}(d\vec{\sigma}_1, ..., d\vec{\sigma}_k; \beta, \varepsilon \mathbf{h}_N) = \tilde{P}_k(d\vec{\sigma}_1, ..., d\vec{\sigma}_k; \beta, \vec{h}),$$

where $\mathbf{h}$ is a homogeneous external field: $(\mathbf{h})_j = \vec{h}$, $j \in \mathbb{N}$. Using, as above, Eqs. (2)–(10) and the Laplace method for the fixed direction $\vec{h}$ one can easily obtain that projections $\{\tilde{P}_k\}_{k=1}^{\infty}$ coincide with the measures $\{P_k\}_{k=1}^{\infty}$; see Eq. (11), for $\vec{\rho} = \vec{h}/h$. Therefore, the quasiaverage method gives us the same result (pure states) as the generalized quasiaverage method in the case (i) (see Proposition 1) or in the case $0 < \alpha < 1$ for the homogeneous external fields (see Remark 3). Let us note that the conventional quasiaverage method corresponds formally to $\alpha = 0$.

Remark 5. We would like, however, to stress that the explicit formula (13) for the mixed states may no longer hold if $\|\mathbf{h}_N^{(N)}\| \nrightarrow 0$ for $N \to \infty$. Let us choose

$$\vec{h}_j^{(N)} = \begin{cases} N^{-1/6}\vec{\rho} & \text{for } 1 \leqslant j < 2\sqrt{N} \\ -2N^{-1/6}\vec{\rho} & \text{for } 2\sqrt{N} \leqslant j < 3\sqrt{N} \\ 0 & \text{for } 3\sqrt{N} \leqslant j \leqslant N \end{cases} \tag{22}$$

Then $\vec{H}_N = 0$, but the third-order term in the expansion in powers of $\vec{h}_j^{(N)}$ in Eq. (14) will remove the degeneracy and lead to a limit state with nonzero magnetization along $\vec{\rho}$.

3. LIMIT STATES IN RANDOM EXTERNAL FIELDS

We shall show here that limit states can be defined and explicitly described for many nonzero external field configurations.

Let $\mathbf{h} = \{\vec{h}_j\}_{j \in \mathbb{N}}$ be a sequence of independent $\mathbb{R}^n$-valued random variables identically distributed according to a probability measure $d\nu(\vec{h})$. Then the random field configurations $\{\mathbf{h}\}$ belong to the probability space $(\Omega, \Sigma, \lambda)$ where we set

$$\Omega = \prod_{j \in \mathbb{N}} [\mathbb{R}^n, d\nu(\vec{h}_j)], \qquad d\lambda(\mathbf{h}) = \prod_{j \in \mathbb{N}} d\nu(\vec{h}_j) \tag{23}$$

Proposition 2. Let the measure $d\nu$ be such that $\int_{\mathbb{R}^n} d\nu(\vec{h})\, |\vec{h}| < \infty$. Then for the Curie–Weiss model (2) with $\vec{h}_j = (\mathbf{h})_j$, $j = 1, 2,..., N$, the limits (4) exist for almost all (with respect to measure $d\lambda$) external field configurations and they are (linear convex) superpositions of

$$\mathbb{P}_k(d\vec{\sigma}_1,..., d\vec{\sigma}_k;\, \vec{y}_0, \mathbf{h}) = \prod_{j=1}^{k} d\vec{\sigma}_j \exp\{\beta[\vec{\sigma}_j \cdot (\vec{y}_0 - \vec{h}_j) - \varphi_n(|\vec{y}_0 - h_j|)]\} \tag{24}$$

where $\vec{y}_0(\beta) \in \mathscr{M}(\beta, \nu)$. Here $\mathscr{M}(\beta, \nu)$ is the set of the minimum points of the function

$$\mathscr{F}_n(\vec{y}) = \tfrac{1}{2} y^2 - \int_{\mathbb{R}^n} d\nu(\vec{h})\, \varphi_n(|\vec{y} - \vec{h}|) \tag{25}$$

Remark 6. In particular, $\{\vec{y}_0(\beta)\}$ are solutions of the following self-consistency equation:

$$\vec{y} = \int_{\mathbb{R}^n} d\nu(\vec{h}) \frac{(\vec{y} - \vec{h})}{|\vec{y} - \vec{h}|} \varphi_n'(|\vec{y} - \vec{h}|) \tag{26}$$

which is useful to compare with Eqs. (6) and (7).

Proof. Note that the shifts act ergodically on Ω. Then for every fixed $\vec{y}$ the quantity

$$\Phi_N(\vec{y}) = \frac{1}{N} \sum_{j=1}^{N} \varphi_n(|\vec{y} - \vec{h}_j|) \tag{27}$$

converges $d\lambda$-almost everywhere ($d\lambda$-a.e.) to $\Phi(y)$ and

$$\Phi(\vec{y}) = \int_{\mathbb{R}^n} dv(\vec{h})\, \varphi_n(|\vec{y}-\vec{h}|) \tag{28}$$

by the individual (or Birkhoff–Khinchine) ergodic theorem.[20] Using convexity of function $\varphi_n(y)$ [see Eq. (5)] one obtains the same for all $\vec{y}$ uniformly on compacts. Moreover, because the quantity $N^{-1}\sum_{j=1}^{N}|\vec{h}_j|$ converges $d\lambda$-a.e. to $\int_{\mathbb{R}^n} dv(\vec{h})\,|\vec{h}|$ the asymptotics at infinity of the function $\Phi_N(\vec{y})$ is uniform. So, all integrals in Eq. (4) can be restricted to a given compact set in evaluating the asymptotics of the density

$$p_k^{(N)}(\vec{\sigma}_1,..., \vec{\sigma}_k; \beta, \mathbf{h}_N) = \int_{\mathbb{R}^n} \omega_{\beta,\mathbf{h}}^{(N)}(d\vec{y}) \prod_{j=1}^{k} \times \exp\{\beta[\vec{\sigma}_j(\vec{y}-\vec{h}_j) - \varphi_n(|\vec{y}-\vec{h}_j|)]\}$$

$$\omega_{\beta,\mathbf{h}}^{(N)}(d\vec{y}) = \frac{\exp\{-\beta N[\frac{1}{2}y^2 - \Phi_N(\vec{y})]\}}{\int_{\mathbb{R}^n} d\vec{y}\exp\{-\beta N[\frac{1}{2}y^2 - \Phi_N(\vec{y})]\}}\, d\vec{y} \tag{29}$$

Using on this compact the uniform ($d\lambda$-a.e.) convergence $\Phi_N(\vec{y}) \to \Phi(\vec{y})$, one obtains the weak ($d\lambda$-a.e.) convergence of the measures $\{\omega_{\beta,\mathbf{h}}^{(N)}(d\vec{y})\}$ in Eq. (29) to the probability measure $\omega_\beta(d\vec{y})$ with support on the set $\mathcal{M}(\beta,v)$ So, with probability 1 with respect to $d\lambda$, we have the following equality:

$$\lim_{N_i\to\infty} P_k^{(N_i)}(d\vec{\sigma}_1,..., d\vec{\sigma}_k; \beta, \mathbf{h}_{N_i}) = \int_{\mathcal{M}(\beta,v)} \omega_\beta(d\vec{y}_0)\, \mathbb{P}_k(d\vec{\sigma}_1,..., d\vec{\sigma}_k; \vec{y}_0, \mathbf{h})$$
$$\equiv \mathcal{P}_k(d\vec{\sigma}_1,..., d\vec{\sigma}_k; \beta, \mathbf{h}) \tag{30}$$

which gives the result by a compactness argument. ▮

Corollary 2. In order to make a connection with the quasiaverage method, it should be noted that if we switch off the external fields by, e.g., scaling the measure $dv(\vec{h})$: $t>0$, $dv_t(\vec{h}) \equiv dv(t^{-1}\vec{h})$ and letting $t\to 0$, then the limit states $d\mathcal{P}_t$ corresponding to configuration $\mathbf{h}$ will converge to a limit state $d\mathcal{P}_0$ corresponding to zero external field. Indeed, according to Eq. (30) the limit state $d\mathcal{P}_0$ is a superposition of the pure states $dP_{\vec{\rho}}$:

$$d\mathcal{P}_0 = \int_{\mathcal{M}(\beta,v_0)} \omega_\beta(d\vec{y}_0)\, dP_{\vec{\rho}(\vec{y}_0)} \tag{31}$$

It is easy to check that Eq. (31) is nothing but another representation of the expression (20) where parameter $\vec{H}$ corresponds to that defined by the limit $v_t \to v_0$ for the case of a homogeneous external field configuration $\mathbf{h}$ see Remark 4.

Remark 7. We do not consider here a very interesting question about the structure of the set $\mathscr{M}(\beta, v)$ [and the limiting set $\mathscr{M}(\beta, v_0)$]. This question is connected with the description of the critical properties of ordered magnetic systems placed in random external fields which are now under active study. For an interesting discussion of Eq. (26) (connected with a spin-glass behavior) we refer to Ref. 21 and also to a recent paper (Ref. 22) about the Ising model in a random magnetic field. We hope to return to this problem elsewhere.

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